

A Constructive Epistemic Logic with Public Announcement (Non-Predetermined Possibilities)

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Abstract

We argue that the notion of epistemic *possible worlds* in constructivism (intuitionism) is not as the same as it is in classic view, and there are possibilities, called non-predetermined worlds, which are ignored in (classic) Epistemic Logic. Regarding non-predetermined possibilities, we propose a constructive epistemic logic and prove soundness and completeness theorems for it. We extend the proposed logic by adding a public announcement operator. To declare the significance of our work, we formulate the well-known Surprise Exam Paradox, **SEP**, via the proposed constructive epistemic logic and then put forward a solution for the paradox. We clarify that the puzzle in the **SEP** is because of students' (wrong) assumption that the day of the exam is necessarily predetermined.

In (classic) epistemic logic (see [4]), the knowledge of an agent is modelled through two fundamental notions

- 1- *possible worlds* (states) and
- 2- *Indistinguishability*.

An agent knows some fact if it is true in all *possible worlds* that the agent cannot *distinguish* them from the actual world.

To propose a Constructive Epistemic Logic, we first emphasize that the notion of *possible worlds* in intuitionism (constructivism) is not as the same as it is in classic view. It is our main idea that leads us to introduce an Epistemic logic from intuitionistic (constructive) point of view.

Our proposed constructive epistemic logic is not much different with (classic) epistemic logic except that it admits a new kind of possible worlds called non-predetermined worlds where the facts of the worlds have not necessarily been determined already!

Suppose I announce on Facebook that I like one and only one of the days of the next week, and on each day, I will write on my Facebook at 10am and announce that whether I like the day or not. Also, suppose that it is Monday 10am now, and I am going to announce my idea about the day. What is the actual world that I am present in? I can announce p : "I like Monday" and also I can announce $\neg p$: "I do not like Monday". That is, in the actual world that I live, it is not determined neither p nor $\neg p$ yet. Kripke models which are defined as semantics of (classic) epistemic logic cannot model the notion of predetermination. For each (classic) epistemic state (M, s) and each atomic formula q , we have either $(M, s) \models q$ or $(M, s) \models \neg q$. Let (N, t) be a (classic) epistemic state which describes my state on Monday.

Then either $(N, t) \models p$ or $(N, t) \models \neg p$, (and certainly not both of them). If $(N, t) \models p$, then I cannot announce “I do not like Monday”, whereas it is up to my free will, and what I like is not necessary predetermined. If $(N, t) \models \neg p$, then I cannot announce “I like Monday”, and I am forced to write “I like Monday”!

On Monday at 10am, both p and $\neg p$ are announce-able and a suitable *epistemic possible world* to describe my state at Monday, should not satisfy neither p nor $\neg p$.

Therefore, it is not possible to describe the actual state that I have on Monday via classic epistemic possible worlds ¹. We need a kind of possible worlds which at them neither p nor $\neg p$ is necessarily predetermined. We use Beth models (a semantic class for constructive logic, see [3]) to obtain this aim.

The paper is organized as follows:

In section 1, we discuss the notion of possibility, and argue that non-predetermination should be considered as a new possibility. We use Beth models (see[3]) to describe non-predetermination possibilities, and then as semantics for our constructive epistemic logic, we introduce a kind of Kripke models, called Beth-Kripke models, where each possible world is a Beth model (instead of a valuation to atomic formula). We then provide an axiomatization system and prove soundness and completeness theorem.

In section 2, we extend our proposed logic, by adding the public announcement operator. The public announcement operator is defined in the way that for those formulas, say φ which their value are not yet determined in the actual world (say (M, s)), both φ and $\neg\varphi$ are announce-able (in other words, we have both $(M, s) \models \langle\varphi\rangle\top$ and $(M, s) \models \langle\neg\varphi\rangle\top$).

The surprise exam paradox, **SEP** (see [8, 9, 10]), was formulated via classic epistemic logic in different ways [2, 9, 11, 13]. Also the paradox was formulated in constructive analysis [1]. In section 3, we formally model **SEP** in the proposed constructive epistemic logic. Then, regarding non-predetermined worlds, we put forward a solution for the paradox.

1 Constructive Possible Worlds

The semantics of classic propositional logic is introduced via valuations of atomic formulas by *True* or *False*. Consider two atomic propositional formulas p and q . Classically, there are exactly four different possible worlds (valuations) for p and q as follows:

- 1- $p = \text{True}, q = \text{True},$
- 2- $p = \text{True}, q = \text{False},$

¹If we want to describe the actual state classically then we need to consider Temporal concepts and thinking of temporal epistemic logics. However, in our work, we show that we can describe non-predetermined cases without using temporal modals and operators.

- 3- $p = \text{False}, q = \text{True},$
- 4- $p = \text{False}, q = \text{False}.$

However, the semantics of constructive (intuitionistic) propositional logic is not the same semantics of classic propositional logic. The semantics of constructive propositional logic is formally introduced by Kripke models or Beth models. In this paper, we consider Beth models (see [3], and chapter 13 of [12]). A Beth model is a triple $\Theta = \langle Q, \leq, F \rangle$, where $\langle Q, \leq \rangle$ is a partially ordered set with the following condition that there exists a node $\alpha \in Q$, called root, such that for all $\beta, \gamma \in Q$, $\alpha \leq \beta$, $\alpha \leq \gamma$ ².

F is mapping assigning atomic formulas to elements of Q . More precisely, let \mathbf{AT} be the set all atomic propositional formulas. Then $F : Q \rightarrow 2^{\mathbf{AT}}$ is a function subject to the following condition: for all $\alpha, \beta \in Q$, if $\alpha \leq \beta$ then $F(\alpha) \subseteq F(\beta)$.

Notation 1.1 *Given a Beth model $\Theta = \langle Q, \leq, F \rangle$, instead of writing $\alpha \in Q$, for simplicity, we write $\alpha \in \Theta$.*

A path P through a node $\alpha \in \Theta$ is a maximal linearly ordered subset of Θ containing α . A bar B for a node $\alpha \in \Theta$ is a subset of Θ with the property that each path through α intersects it.

The satisfaction relation $\Vdash \subseteq \Theta \times \text{SENT}(\mathbf{AT})$ (where $\text{SENT}(\mathbf{AT})$ is the set of all propositional formulas over \mathbf{AT}), is defined inductively,

- $\alpha \Vdash p$ iff there is a bar B for α such that for each $\beta \in B$, $p \in F(\beta)$, (for atomic $p \in \mathbf{AT}$).
- $\alpha \Vdash A \wedge B$ iff $\alpha \Vdash A$ and $\alpha \Vdash B$.
- $\alpha \Vdash A \vee B$ iff there is a bar B for α , such that for each $\beta \in B$, $\beta \Vdash A$ or $\beta \Vdash B$.
- $\alpha \Vdash A \rightarrow B$ iff for each $\beta \geq \alpha$, $\beta \Vdash A$ then $\beta \Vdash B$.
- $\alpha \Vdash \neg A$ iff for each $\beta \geq \alpha$, $\beta \not\Vdash A$.

We say two Beth models (Θ_1, α_1) and (Θ_2, α_2) are equivalent whenever they satisfy the same propositional formulas.

Theorem 1.2 *Given a Beth model (Θ, α) , for any propositional formula A ,*

- a. $\alpha \Vdash A$ iff there is a bar B for α such that for all $\beta \in B$, $\beta \Vdash A$.
- b. $\alpha \not\Vdash A$ iff there is a path P through α such that for each $\beta \in P$, $\beta \not\Vdash A$.
- c. $\alpha \leq \beta$ and $\alpha \Vdash A$ then $\beta \Vdash A$.

Proof. See [3]. \dashv

Let Γ be a set of propositional formulas. By $\Gamma \vdash_i A$ we mean A is derivable in constructive propositional logic. By $\Gamma \models A$, we mean A is satisfied in all Beth models which satisfy all formulas in Γ .

²This condition is extra, and we consider it here for convenience.

Theorem 1.3 *Soundness and Completeness Theorem:* $\Gamma \vdash_i A$ iff $\Gamma \models A$.

Proof. See [3]. \dashv

In the beginning of the section, we mentioned that there are four different possible worlds for two atomic formulas p and q in classic view. In constructive propositional logic, regarding Beth models (instead of valuations), the number of possible worlds are more than four.

Let $\Gamma = \{p \vee q, \neg(p \wedge q)\}$. Classically, two possible worlds are considerable for Γ

c1- $p = \text{True}, q = \text{False}$, and

c2- $p = \text{False}, q = \text{True}$.

But there are three non-equivalent Beth models for Γ ,

i1- (Θ, α) , where $\Theta = \{\alpha\}$, and $F(\alpha) = \{p\}$,

i2- (Θ', α') , where $\Theta' = \{\alpha'\}$, and $F(\alpha') = \{q\}$, and

i3- (Θ'', α'') , where $\Theta'' = \{\alpha'', \beta'', \gamma''\}$, $F(\alpha'') = \emptyset$, $F(\beta'') = \{p\}$, $F(\gamma'') = \{q\}$, $\alpha'' \leq \beta''$, and $\alpha'' \leq \gamma''$.

The two cases $i1$ and $i2$ are the same classical cases $c1$ and $c2$, but the third one, $i3$, is new. The possible world (Θ'', α'') is a situation where formulas $p \vee q, \neg(p \wedge q)$ holds true but neither p nor q are *predetermined*. The possibility of non-predetermination is regarded in Beth models, whereas classically, it is presupposed that valuation of atomic formulas are predetermined already.

Recalling the Surprise Exam Paradox (see section 3), suppose p stands for "Tomorrow, the teacher will take the exam", and q stands for "the teacher will take the exam the day after tomorrow"

- 1- The Beth model (Θ, α) represents the possible world where it is already determined that teacher will take the exam tomorrow.
- 2- The Beth model (Θ', α') represents the possible world where it is already determined that teacher will take the exam the day after tomorrow.
- 3- The Beth model (Θ'', α'') represents the possible world where the teacher has not already decided whether take the exam tomorrow or one day later. Classically, we cannot represent the third possibility.

One may assume the root of the Beth model (Θ'', α'') as the current state of the possible world. Other nodes with respect to partial order relation are future nodes of the current state³. Future is not predetermined and it is the teacher who determines it later by his *free will*. It is up to free will of the teacher to take the exam tomorrow or not, and neither the value of p nor the value of $\neg p$ is not determined already. Beth models help us describe *non-predetermination* as a new possibility. The *non-predetermination* is ignored in (classic) epistemic logic.

³The reader may note that to regard future, we can also argue in terms of temporal logic. But constructive (intuitionistic) logic, without having temporal modal operators, in some sense, considers this case. Therefore, avoiding extra modal operators, we propose a constructive epistemic logic.

Definition 1.4 Let \mathbf{AT} be a non-empty set of propositional variables, and \mathcal{A} be a set of agents. The language $L(\mathcal{A}, \mathbf{AT})$ is the smallest superset of \mathbf{AT} such that

$$\text{if } \varphi, \psi \in L(\mathcal{A}, \mathbf{AT}) \text{ then } \neg\varphi, (\varphi \wedge \psi), (\varphi \vee \psi), \varphi \rightarrow \psi, K_i\varphi \in L(\mathcal{A}, \mathbf{AT}),$$

for $i \in \mathcal{A}$.

For $i \in \mathcal{A}$, $K_i\varphi$ has to be read as ‘agent i knows φ ’.

Definition 1.5 A Beth-Kripke model M is a tuple $M = \langle S, (\sim_i)_{i \in \mathcal{A}} \rangle$, where S is a non-empty set of Beth models over \mathbf{AT} as possible worlds, (each $s \in S$ is a pointed Beth model (Θ_s, α_s) where α_s is the root of Θ_s), and each \sim_i is a binary accessibility relation between worlds.

Let $M = \langle S, (\sim_i)_{i \in \mathcal{A}} \rangle$ be a Beth-Kripke model, and Θ be an arbitrary Beth model of M (i.e., $\Theta = \Theta_s$ for some $s \in S$). We define the satisfaction relation $\Vdash_M \subseteq \Theta \times L(\mathcal{A}, \mathbf{AT})$ as follows:

- for all $\varphi \in SENT(\mathbf{AT})$, for each $\alpha \in \Theta$, $\alpha \Vdash_M \varphi$ iff $\alpha \Vdash \varphi$.
- For each $\alpha \in \Theta$, for each agent $i \in \mathcal{A}$, $\alpha \Vdash_M K_i\varphi$ iff for all $\Omega \in S$, if $\Theta \sim_i \Omega$, then for all $\beta \in \Omega$, $\beta \Vdash_M \varphi$.
- For each $\alpha \in \Theta$, $\alpha \Vdash_M \varphi \wedge \psi$ iff $\alpha \Vdash_M \varphi$ and $\alpha \Vdash_M \psi$.
- For each $\alpha \in \Theta$, $\alpha \Vdash_M \varphi \vee \psi$ iff there is a bar $B \subseteq \Theta$ for α , such that for each $\beta \in B$, $\beta \Vdash_M \varphi$ or $\beta \Vdash_M \psi$.
- For each $\alpha \in \Theta$, $\alpha \Vdash_M \varphi \rightarrow \psi$ iff for each $\beta \geq_\Theta \alpha$, $\beta \Vdash_M \varphi$ then $\beta \Vdash_M \psi$.
- $\alpha \Vdash_M \neg\varphi$ iff for each $\beta \geq_\Theta \alpha$, $\beta \nVdash_M \varphi$.

Theorem 1.6 Let $M = \langle S, (\sim_i)_{i \in \mathcal{A}} \rangle$ be a Beth-Kripke model, and Θ be an arbitrary Beth model of M . For each $\alpha \in \Theta$, and $\varphi \in L(\mathcal{A}, \mathbf{AT})$, we have

- a. $\alpha \Vdash_M \varphi$ iff there is a bar B for α such that for all $\beta \in B$, $\beta \Vdash_M \varphi$.
- b. $\alpha \nVdash_M \varphi$ iff there is a path P through α such that for each $\beta \in P$, $\beta \nVdash_M \varphi$.
- c. $\alpha \leq \beta$ and $\alpha \Vdash_M \varphi$ then $\beta \Vdash_M \varphi$.
- d. $\alpha \Vdash_M K_i\varphi$ then for all $\beta \in \Theta$, $\beta \Vdash_M K_i\varphi$.

Proof. It is straightforward. \dashv

Definition 1.7 Let $\varphi \in L(\mathcal{A}, \mathbf{AT})$ and $M = \langle S, (\sim_i)_{i \in \mathcal{A}} \rangle$ be a Beth-Kripke model. We say (M, s) satisfies φ , denoted by $(M, s) \models \varphi$, whenever the root of Θ_s satisfies the formula φ regarding the model M , i.e., $\alpha_s \Vdash_M \varphi$.

Theorem 1.8 Let $M = \langle S, (\sim_i)_{i \in \mathcal{A}} \rangle$, $s \in S$, and $i \in \mathcal{A}$ be an arbitrary agent. Also let $\Gamma = \{\varphi \in \text{SENT}(\mathbf{AT}) \mid (M, s) \models K_i \varphi\}$. Then Γ is a constructive (intuitionistic) propositional closed theory. That is, for every $\psi \in \text{SENT}(\mathbf{AT})$, $\psi \in \Gamma$, iff ψ is derivable from Γ in constructive propositional logic, $\Gamma \vdash_i \psi$.

Proof. Suppose $\Gamma \vdash_i \varphi$ in constructive (intuitionistic) logic. Then by soundness and completeness theorem 1.3, φ is satisfied in all Beth models which satisfy all formulas in Γ . Hence φ is satisfied in all Beth model (Θ_t, α_t) in model M , which $t \sim_i s$. Therefore, by definition of satisfaction for knowledge, we have $(M, s) \models K_i \varphi$, and we are done. \dashv Above theorem

declares that

knowing a fact in constructive point of view is to have a proof for the fact.

Theorem 1.9 Let $M = \langle S, (\sim_i)_{i \in \mathcal{A}} \rangle$, $s \in S$, $i \in \mathcal{A}$, and $\varphi \in L(\mathcal{A}, \mathbf{AT})$. Then $(M, s) \models K_i \varphi \vee \neg K_i \varphi$.

Proof. By definition 1.7, $(M, s) \models K_i \varphi \vee \neg K_i \varphi$ is equivalent to $\alpha_s \Vdash_M K_i \varphi \vee \neg K_i \varphi$. Note that, according to item d) of theorem 1.6, for all $\beta \in \Theta_s$, if $\beta \Vdash_M K_i \varphi$ then for all $\delta \in \Theta_s$, we have $\delta \Vdash_M K_i \varphi$. Therefore, Either none of the nodes of the Beth model Θ satisfies $K_i \varphi$, and consequently by definition 1.7, we have $\alpha_s \Vdash_M \neg K_i \varphi$. Or all nodes of the Beth model Θ satisfies $K_i \varphi$, and consequently, $\alpha_s \Vdash_M K_i \varphi$. Thus, we have $\alpha_s \Vdash_M K_i \varphi \vee \neg K_i \varphi$. \dashv The

above theorem says, knowing is decidable. Intuitionistic (constructive) propositional logic, **IPC** (see [12], chapter 2), is effectively decidable. That is, it is decidable that whether a formula is a theorem or not. As the notion of knowledge in intuitionism is considered equal to having evidence (proof), it is plausible that for an appropriate constructive epistemic logic knowing is decidable. The two above theorems justifies why we call our proposed epistemic logic as a constructive epistemic logic.

1.1 Axiomatization

In this part, we introduce an axiomatization system dented by **IS6**, and prove soundness and complexness theorem with respect to $S5$ Beth-Kripke models ⁴. The logic of **IS6** is much similar to the $S5$ epistemic logic introduced in [4], pages 26-29.

The constructive epistemic logic $IS6$ consists of axioms $A1 - A6$ and the derivation rules $R1$ and $R2$ given below

$R1: \vdash \varphi, \vdash \varphi \rightarrow \psi \Rightarrow \vdash \psi$

$R2: \vdash \varphi \Rightarrow K_i \varphi, \text{ for all } i \in \mathcal{A}$

$A1: \text{Axioms of constructive propositional logic}$

$A2: (K_i \varphi \wedge K_i(\varphi \rightarrow \psi)) \rightarrow K_i \psi$

$A3: K_i \varphi \rightarrow \varphi$

$A4: K_i \varphi \rightarrow K_i K_i \varphi$

$A5: \neg K_i \varphi \rightarrow K_i \neg K_i \varphi$

$A6: \neg K_i \varphi \vee K_i \varphi$

⁴ A Beth-Kripke model is $S5$ whenever the relations \sim_i are reflexive, transitive, and Euclidian.

Theorem 1.10 (*Soundness and Completeness*). *Axiom system IS6 is sound and complete with respect to semantic class of S5 Beth-Kripke models.*

Proof. \dashv

2 A Constructive Epistemic Public Announcement Logic

We extend the proposed logic by adding public announcement operator to construct a constructive epistemic public announcement logic, **CEPAL** similar to the (classic) public announcement logic [4], page 73,

2.1 Syntax

Given a finite set of agents \mathcal{A} , and a set of atomic formulas **AT**, the language of $L_{IEPAL}(\mathcal{A}, \mathbf{AT})$ is inductively defined by the BNF:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \psi, \varphi \vee \psi \mid \varphi \rightarrow \psi \mid K_i\varphi \mid [\varphi]\psi$$

where $i \in \mathcal{A}$, and $p \in \mathbf{AT}$.

2.2 Semantics

We describe public announcement operator on Beth-Kripke models. Let $M = \langle S, (\sim_i)_{i \in \mathcal{A}} \rangle$ be a Beth-Kripke model. Let (Θ_s, α_s) be a Beth model of the model M . For a formula φ , we define the Beth model $\Theta_s|_\varphi$ as follows:

- $\Theta_s|_\varphi = \{\alpha \in \Theta_s \mid \alpha \not\models_M \neg\varphi\}$,
- the partial order relation of the Beth model $\Theta_s|_\varphi$ is obtain by restriction of partial order relation of the model Θ_s to the set of nodes in $\Theta_s|_\varphi$.

We let $M|_\varphi = \langle S', (\sim'_i)_{i \in \mathcal{A}} \rangle$ with

$$S' = \{(\Theta_s|_\varphi, \alpha_s) \mid s \in S \text{ \& } (M, s) \not\models \neg\varphi\}, \text{ and}$$

$$\sim'_i = \sim_i \cap (S' \times S').$$

For each $\beta \in \Theta_s$, we define $\beta \Vdash_M [\varphi]\psi$, if and only if for α_s (the root of Θ_s) we have $\alpha_s \not\models_M \neg\varphi$ and $(\Theta_s|_\varphi, \alpha_s) \Vdash_{M|_\varphi} \psi$. We then define

$$(M, s) \models [\varphi]\psi \text{ iff } (M, s) \not\models \neg\varphi \text{ implies } (M|_\varphi, s) \models \psi$$

Where

The dual of $[\varphi]$ is $\langle\varphi\rangle$:

$$(M, s) \models \langle\varphi\rangle\psi \text{ iff } (M, s) \not\models \neg\varphi \text{ and } (M|_\varphi, s) \models \psi$$

Proposition 2.1 *Let $M = \langle S, (\sim_i)_{i \in \mathcal{A}} \rangle$ be a Beth-Kripke, and (Θ, α) be a Beth model of the model M . For each formula φ , for each $\beta \in \Theta|_\varphi$, we have $\beta \Vdash \langle\varphi\rangle\psi$.*

Proof. It is straightforward. \dashv

Announcing a formula φ , two kinds of updates happen on a Beth-Kripke model

I. Updating each possible world (Beth model) of the model:

In a constructive possible world (a Beth model (Θ, α)), some matters of the world are predetermined and some others are not. When a formula φ is announced, all nodes of the Beth model Θ , say $\gamma \in \Theta$, which it is impossible that φ gets determined in future (i.e., for all $\beta \geq \gamma$, $\beta \not\models_M \varphi$, and thus $\beta \Vdash_M \neg\varphi$) are removed.

II. Updating the indistinguishability relations of agents:

The set of epistemic possible worlds is restricted to those possible worlds which it is possible that φ gets determined, and indistinguishability relations is the same relation regarding remained possible worlds.

In (classic) public announcement logic, *PAL* (see [4], chapter 4), for a classic possible world (M, s) , and a formula φ , only one of the two formulas φ and $\neg\varphi$ is *announcable* (an executable announcement). It is because that in a classic possible world either φ is true or $\neg\varphi$, and not both of them, and to announce a formula, the formula must be already true in the actual world. Whereas, in our Beth-Kripke models, both φ and $\neg\varphi$ could be *announcable*! It is because that in a Beth model it is possible that neither $\neg\varphi$ nor $\neg\neg\varphi$ are satisfied.

For example, assume the Beth model (Θ, α) , where $\Theta = \{\alpha, \beta, \gamma\}$, $F(\alpha) = \emptyset$, $F(\beta) = \{p\}$, $F(\gamma) = \{q\}$, $\alpha \leq \beta$, and $\alpha \leq \gamma$. Let M be a Beth-Kripke model with just one world s , where $\Theta_s = \Theta$, and $\alpha_s = \alpha$, and $s \sim_i s$, for all $i \in \mathcal{A}$. Both formula p and $\neg p$ are announcable (executable announcements) in (M, s) since $(M, s) \not\models p$ and $(M, s) \not\models \neg p$, and thus $(M, s) \models \langle p \rangle \top \wedge \langle \neg p \rangle \top$.

The Beth-Kripke model $M|_p$ is a model with the world $(\Theta|_p, \alpha)$ which has two nodes α, β , and $F(\beta) = \{p\}$, $F(\alpha) = \emptyset$. We have $(M|_p, s) \models \neg q$, and thus $(M, s) \models [p]\neg q$.

The Beth-Kripke model $M|_{\neg p}$ is a model with world $(\Theta|_{\neg p}, \alpha)$ which has two nodes α, γ , and $F(\alpha) = \emptyset$, $F(\gamma) = \{q\}$. We have $(M|_{\neg p}, s) \models q$, and thus $(M, s) \models [\neg p]q$.

In (classic) public announcement logic, *PAL* (see [4], chapter 4), it is possible to translate each formula in the language of public announcement logic to an equivalent formula in the language of epistemic logic without public announcement (see proposition 4.22, [4]). In contrast, in our proposed public announcement logic, there are formulas which are not equivalent to any formula without announcement operator.

Proposition 2.2 *Let $\mathbf{AT} = \{p\}$. The formula $\langle p \rangle \top$ is not equivalent to any propositional formula $\varphi \in \text{SENT}(\mathbf{AT})$.*

Proof. Assume $\langle p \rangle \top$ is equivalent to a propositional formula $\varphi \in \text{SENT}(\mathbf{AT})$. Then for every pointed Beth model (Θ, α) , $(\Theta, \alpha) \Vdash \langle p \rangle \top \leftrightarrow \varphi$. Consider the following Beth model (Θ, α) , where $\Theta = \{\alpha, \beta, \gamma\}$, $F(\alpha) = \emptyset$, $F(\beta) = \{p\}$, $F(\gamma) = \emptyset$, $\alpha \leq \beta$, and $\alpha \leq \gamma$. We have $\alpha \Vdash \langle p \rangle \top$, by our assumption, we have $\alpha \Vdash \varphi$. By item c) of theorem 1.2, $\gamma \Vdash \varphi$. Since $F(\gamma) = \emptyset$, we have φ is a tautology, i.e., $\varphi \leftrightarrow \top$. Hence, we have $\langle p \rangle \top$ is a tautology, i.e., $\langle p \rangle \top$ is satisfied in all Beth models. Contradiction. \dashv

Theorem 2.3 *Regarding finite S5 Beth-Kripke models, the followings is valid, for every $\varphi, \psi \in L_{IEPAL}(\mathcal{A}, \mathbf{AT})$, and $p \in \mathbf{AT}$.*

$$[\varphi]\psi \leftrightarrow (\varphi \rightarrow \psi)$$

Proof. \dashv

3 The Surprise Exam Paradox

We formulate the well-known Surprise Exam Paradox, **SEP**, in our constructive epistemic public announcement logic. The **SEP** is as follows:

The teacher announces to the students:

you will have one and only one exam at 10am on one day in the last third days of the next week (Wednesday-Thursday-Friday), but you will not know in advance the day of exam.

The students, using a *backward argument*, reason that *there can be no exam* indeed:

- Friday is not the day of the exam. Since if it is, we will not have received the exam by Thursday night, and as there is an exam in the week, then at Thursday night, we will be able to know in advance the day of exam,
- Thursday is not the day of the exam. Since if it is, we will not have received the exam by Wednesday night, and as there is an exam in the week, and it is not on Friday, then at Wednesday night, we will know in advance the day of exam is on on Thursday,...
- and Wednesday is not the day of the exam by a similar argument,
- then none of the days is the day of exam.

The paradox was investigated in terms of (classic) epistemic notions by several people [2, 9, 11, 13]. For example, Gerbrandy sees the puzzle in the assumption that announcements are in general successful [5], and Baltag, to solve the paradox, lets the students to revise their trust to the teacher once they reach the paradox [7]. In [1], the paradox is investigated in a constructive view, considering *free will* of the teacher. In this paper, we claim that

the puzzle in **SEP** is that students (wrongly) assume the day that teacher is going to take the exam is predetermined!

Consider the following version of **SEP** which is obviously equivalent to the standard version: The teacher announces to the students:

I like one and only one of the days among Wednesday, Thursday, and Friday. I start to announce one by one whether I like the days or not beginning with Wednesday, then Thursday, and finally Friday. You will not know, in advance, the day I like before I announce that I like that day.

Let $AT = \{p_1, p_2, p_3\}$ where

p_1 stands for "I like Wednesday",

p_2 stands for "I like Thursday", and

p_3 stands for "I like Friday".

It is up to desire of the teacher to like which day, and his desire could be non-predetermined. Therefore, the teacher can choose freely announce either p_1 or $\neg p_1$ (certainly, not both of them together). But, in a classic epistemic possible world either p_1 is true or $\neg p_1$, and thus just one of the formulas p_1 and $\neg p_1$ is an executable-announcement, and thus classic epistemic possible worlds are not suitable to formulate the paradox.

When the teacher uses the term 'like', he means that he announces up to his free will, and students must assume that both p_1 and $\neg p_1$ are executable-announcements at the beginning, and if the teacher announces $\neg p_1$ then both p_2 and $\neg p_2$ are executable-announcements, and finally if the teacher first announces $\neg p_1$ then announces $\neg p_2$, at the end he can announce p_3 , or in other words, p_3 is an executable announcements. Therefore, more formally, if (M, s) is a (constructive) epistemic possible world which represents the situation of the **SEP** then we have

$$(M, s) \models \langle p_1 \rangle \top \wedge \langle \neg p_1 \rangle \top \wedge \langle \neg p_1 \rangle \langle p_2 \rangle \top \wedge \langle \neg p_1 \rangle \langle \neg p_2 \rangle \top \wedge \langle \neg p_1 \rangle \langle \neg p_2 \rangle \langle p_3 \rangle \top.$$

Let us formally describe teacher's announcement in the language of our proposed logic.

Claims0 The teacher claims that he likes one and only one of the days among Wednesday, Thursday, and Friday:

$$\varphi_0 := (p_1 \vee p_2 \vee p_3) \wedge \neg(p_1 \wedge p_2) \wedge \neg(p_1 \wedge p_3) \wedge \neg(p_2 \wedge p_3).$$

The teacher says: "I start to announce one by one whether I like the days or not beginning by Wednesday, then Thursday, and finally Friday. You will not know, *in advance*, the day I like before I announce that I like that day".

He means that at least *one* of the three following claims is true.

claim1) I (the teacher) can announce that I like Wednesday, and you (students) would not know that I like Wednesday before I announce it:

$$\varphi_1 := \langle p_1 \rangle \top \wedge \neg K p_1.$$

claim2) I can announce that I do not like Wednesday, and after that I can announce that I like Thursday, but you would not know that I like Thursday before I announce it

$$\varphi_2 := \langle \neg p_1 \rangle \langle p_2 \rangle \top \wedge \langle \neg p_1 \rangle \neg K p_2.$$

claim3) I can announce that I do not like Wednesday, and after that I can announce that I do not like Thursday, and finally I can announce that I like Friday, but you would not know that I like Friday before I announce it

$$\varphi_3 := \langle \neg p_1 \rangle \langle \neg p_2 \rangle \langle p_3 \rangle \top \wedge \langle \neg p_1 \rangle \langle \neg p_2 \rangle \neg K p_3.$$

In this way the teacher's announcement is

$$\varphi_0 \wedge (\varphi_1 \vee \varphi_2 \vee \varphi_3).$$

The following Beth-Kripke model $M = \langle S, (\sim_i)_{i \in \mathcal{A}} \rangle$ represents the The (constructive) epistemic world of **SEP**, with

$$S = \{s\},$$

$$\mathcal{A} = \{student\},$$

$$\sim_{student} = \{(s, s)\},$$

where (Θ_s, α_s) is defined as follows: $\Theta_s = \{\alpha_s, \beta, \gamma, \delta\}$, $F(\alpha_s) = \emptyset$, $F(\beta) = \{p_1\}$, $F(\gamma) = \{p_2\}$, $F(\delta) = \{p_3\}$, and $\alpha_s \leq \beta$, $\alpha_s \leq \gamma$, $\alpha_s \leq \delta$.

One may easily verify that $(M, s) \models \varphi_0 \wedge (\varphi_1 \vee \varphi_2 \vee \varphi_3)$.

The students reasons that the third claim, φ_3 , is not true as the teacher likes at least one of the days, and after he announced that he does not like Wednesday and Thursday, the student deduce that he must like Friday. However, students cannot apply a backward argument in constructive epistemic logic.

The student correctly argues that the assumption of φ_3 leads to contradiction, and we also have $(M, s) \models \neg\varphi_3$. Then they again correctly derives from $(M, s) \models \neg\varphi_3$ that $(M, s) \not\models p_3$.

In classic epistemic logic negation of a formula is defined by non-satisfaction of the formula (i.e., $((N, t) \models \neg\psi$ iff $(N, t) \not\models \psi$), thus students derive $(M, s) \models \neg p_3$, and using $\neg p_3$, by a backward argument, they derive $\neg p_2$, and after that $\neg p_1$.

However, in Beth-Kripke model, we *cannot* derive from $(M, s) \not\models p_3$ that $(M, s) \models \neg p_3$. The atomic formula p_3 is not predetermined at the node (Θ_s, α_s) , and we have $\alpha_s \not\models p_3$, and $\alpha_s \not\models \neg p_3$. In this way, the student's reasoning does not work in constructive epistemic logic with public announcement.

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